Counting 1324-avoiding Permutations

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Abstract

We consider permutations that avoid the pattern 1324. By studying the generating tree for such permutations, we obtain a recurrence formula for their number. A computer program provides data for the number of 1324-avoiding permutations of length up to 20.

1 Introduction

Let S_n denote the set of all permutations of length n. A permutation $\pi = (p_1, p_2, \ldots, p_n) \in S_n$ contains a pattern $\tau = (t_1, t_2, \ldots, t_k) \in S_k$ if there is a sequence $1 \leq i_{t_1} < i_{t_2} < \cdots i_{t_k} \leq n$ such that $p_{i_1} < p_{i_2} < \cdots < p_{i_k}$. A permutation π avoids a pattern τ, in other words π is τ-avoiding, if π does not contain τ. We write $S_n(\tau)$ for the set of all τ-avoiding permutations of length n, and $s_n(\tau)$ for the cardinality of $S_n(\tau)$. Patterns τ_1 and τ_2 are *Wilf-equivalent* if $s_n(\tau_1) = s_n(\tau_2)$ [Wil02]. A permutation π is $\{\tau_1, \tau_2, \ldots, \tau_n\}$-avoiding if π does not contain any of the patterns from the set.

It is a natural and easy-looking question to ask for the exact formula for $s_n(\tau)$. However, this problem turns out to be very difficult. Although a lot of results on this and related problems have been discovered in the last thirty years, exact answers are only known in a few cases. For all patterns τ of length 3, $s_n(\tau) = C_n$ [Knu73], where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number, a classical sequence [Sta99]. When τ is of length 4, it has been shown that the only essentially different patterns are 1234, 1342 and 1324; all other patterns of length 4 are Wilf-equivalent to one of these three [Sta94, Sta96, BW00]. Regev [Reg81] showed that $s_n(1234)$ asymptotically equals $c\frac{9^n}{n^4}$, where c is a constant given by a multiple integral. Gessel [Ges90] later used theory of symmetric functions to give a generating function for 1234-avoiding permutations. Bóna [Bón97a] enumerated

1342-avoiding permutations, giving their ordinary generating function:

$$\sum_n s_n(1342)x^n = \frac{32x}{-8x^2+20x+1-(1-8x)^{3/2}}.$$

However, the exact enumeration of 1324-avoiding permutations is still an outstanding open problem that we address in this paper.

The problem of avoiding more than one pattern was first studied by Simion and Schmidt [SS85], who determined the number of permutations avoiding two or three patterns of length 3. The numbers of permutations avoiding certain pairs of patterns of length 4 give the Schröder numbers [Wes95]. West [Wes96] also used *generating trees* [CGHK78] to enumerate permutations avoiding all pairs of a pattern of length 3 and a pattern of length 4. Recently, Albert et al. [AAA+03] enumerated {1324, 31524}-avoiding permutations, while finding connections with queue jumping.

We provide a full characterization for the generating tree of 1324-avoiding permutations. This result, combined with a simple computer program, provides data for $s_n(1324)$ for n up to 20. In particular, we show the following:

Theorem 1. *The number $s_n(1324)$ of 1324-avoiding permutations of length n is $g(\langle 1\rangle, n)$, where g is determined by the following recursive formula:*

$$g(\langle a_1 \dots a_m\rangle, n) = \begin{cases} \sum_{i=1}^{m} a_i & \text{if } n = 1, \\ \sum_{i=1}^{m} g(f(\langle a_1 \dots a_m\rangle, i), n-1) & \text{if } n > 1 \end{cases} \tag{1}$$

and $f(\langle a_1 \dots a_m\rangle, i) = \langle b_1 \dots b_{a_i}\rangle$, where:

$$b_j = \begin{cases} a_i + 1 & \text{if } j = 1, \\ \min(i+1, a_j) & \text{if } 2 \le j \le i, \\ a_{j-1} + 1 & \text{if } i < j \le a_i. \end{cases} \tag{2}$$

We conclude by enumerating 1324-avoiding permutations in a specific *strong* class, which is conjectured to be the largest.

2 Proof of Theorem 1

We apply generating trees to count 1324-avoiding permutations. First, we briefly describe succession rules and generating trees. They were introduced in [CGHK78] for the study of Baxter permutations and further applied to the study of pattern-avoiding permutations by Stankova and West [Sta94, Sta96, Wes95, Wes96]. Recently, Barcucci et al. developed ECO [BDLPP99], a methodology for the enumeration of combinatorial objects, which is based on the technique of generating trees.

Definition 2. A generating tree is a rooted, labelled tree such that the labels of the set of children of each node v can be determined from the label of v itself. In other words, a generating tree can be specified by a recursive definition consisting of:

1. **basis:** the label of the root
2. **inductive step:** a set of succession rules that yields a multiset of labelled children depending solely on the label of the parent.

Given $\pi = (p_1, p_2, \ldots, p_n) \in S_n$, we call the position to the left of p_1 position 0, the position between p_i and p_{i+1}, where $1 \leq i \leq n-1$, position i, and the position to the right of p_n position n. We will refer to any of these positions as a *site* of π.

Definition 3. Let τ be a forbidden pattern. The position i, $0 \leq i \leq n$, of a permutation $\pi \in S_n(\tau)$ is an *active site* if inserting $n+1$ into position i gives a permutation belonging to the set $S_{n+1}(\tau)$; otherwise it is said to be an *inactive site*.

Following the methodology developed in [Wes96, Wes95], the generating tree for τ-avoiding permutations is a rooted tree whose nodes on level n are exactly the elements of $S_n(\tau)$. The children of a permutation π of length $n-1$ are all the τ-avoiding permutations obtained by inserting n into π. Each node in the tree is assigned a label; in the simplest case, the label is the number of active sites of π. Typical applications of generating trees analyze changes in the number of active sites after inserting n in a permutation of length $n-1$. These changes determine the labels in the tree and the list of succession rules. Our application considers one more step: to keep the label of every node completely determined from the label of its parent, we consider the changes after inserting n and also $n+1$.

Given a node π at level $n-1$ in the generating tree for 1324-avoiding permutations, let π_n^i be π's children obtained by inserting n into the i-th active site of π. The label assigned to π_n^i is the pair $(s(\pi), i)$, where the sequence $s(\pi) = \langle l(\pi_n^1) \ldots l(\pi_n^{l(\pi)}) \rangle$ contains the number of active sites $l(\pi_n^j)$ for all children π_n^j of π, i.e., for π_n^i and all its siblings. The following completely characterizes this generating tree.

Lemma 4. *All* 1324*-avoiding permutations of length* n *lie on the* n*-th level of the generating tree (Figure 1) defined by the following succession rules:*

$$\begin{cases} \textit{basis:} & (\langle 2 \rangle, 1) \\ \textit{inductive step:} & (\langle a_1 \ldots a_m \rangle, i) \to (\langle b_1 \ldots b_{a_i} \rangle, a_i)(\langle b_1 \ldots b_{a_i} \rangle, a_i - 1) \ldots (\langle b_1 \ldots b_{a_i} \rangle, 1) \end{cases}$$

where $\langle b_1 \ldots b_{a_i} \rangle = f(\langle a_1 \ldots a_m \rangle, i)$ *as in* (2).

Proof. First, we make the following observation. Given a 1324-avoiding permutation $\pi = (p_1, p_2, \ldots, p_{n-1})$ of length $n-1$, the active sites of π are actually the first $l(\pi)$ sites; we can order 132 patterns in π by the occurrence of their 2 and n can be inserted anywhere to the left of the first 2, but nowhere to the right of it.

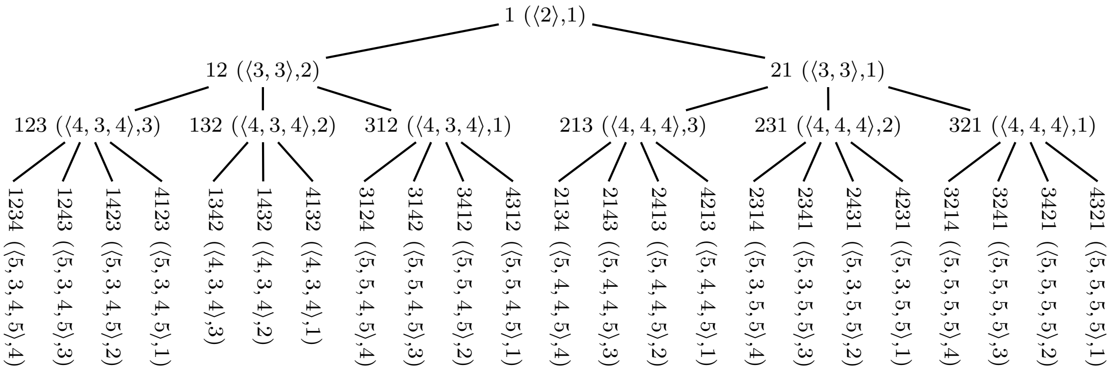

Figure 1: The generating tree for 1324-avoiding permutations

Inserting n into the i-th active site of π certainly creates one new active site in π_n^i, since $n+1$ can be inserted into π_n^i right in front and right behind n. However, inserting n into π may deactivate some active sites in π, because n can play a role of 3 for some 132 pattern in π_n^i that was not in π. In other words, if we order 132 patterns in π and π_n^i by the occurrence of their 2, the first 2 in π_n^i may be to the left of the first 2 in π. The index of the first 2 that n introduces in π_n^i is $\min\limits_{k>i-1, p_k>\min(p_1,p_2,\ldots,p_{i-1})} k$. Since the active sites of π_n^i are exactly the sites to the left of the first 2, the number of active sites in π_n^i is:

$$l(\pi_n^i) = 1 + \min\{l(\pi), \min_{k>i-1, p_k>\min(p_1\ldots p_{i-1})} k\} \tag{3}$$

Notice that $l(\pi_n^i) > i$, since $l(\pi) \geq i$ and $k \geq i$.

In the special case when $i = 1$, i.e., when π_n^i starts with n, we have $l(\pi_n^1) = 1 + l(\pi)$, since n cannot play the role of 3 for any 132 pattern. In general, however, the equation (3) does not express $l(\pi_n^i)$ solely in terms of $l(\pi)$. This is why we consider the next step, inserting $n+1$ into π_n^i.

Let $\pi_{n,n+1}^{i,j}$ be the permutation obtained by inserting $n+1$ into the j-th active site of π_n^i (which is not necessarily the j-th active site of π). We do a case analysis based on j; in each of three cases, the position of the first 2 is the key of our analysis:

- $j = 1$

  Then $\pi_{n,n+1}^{i,j}$ starts with $n+1$ and $l(\pi_{n,n+1}^{i,j}) = 1 + l(\pi_n^i)$.

- $2 \leq j \leq i$

  Then $n+1$ is inserted to the left of n and we have

  $$\pi_{n,n+1}^{i,j} = (p_1, \ldots, p_{j-1}, n+1, p_j, \ldots, p_{i-1}, n, p_i, \ldots, p_{n-1})$$

  Hence, $\pi_{n,n+1}^{i,j}$ has a 132 pattern where any element to the left of $n+1$ serves as 1, $n+1$ serves as 3, and n serves as 2. Thus, n may be the first 2 in $\pi_{n,n+1}^{i,j}$. Further, the number of active sites in $\pi_{n,n+1}^{i,j}$ equals the number of active sites in

$\pi_n^j = (p_1, \ldots, p_{j-1}, n, p_j, \ldots, p_{n-1})$, unless n is the first 2 in $\pi_{n,n+1}^{i,j}$, which reduces the number of active sites in $\pi_{n,n+1}^{i,j}$ to the index of entry n. Therefore, $l(\pi_{n,n+1}^{i,j}) = \min(i+1, l(\pi_n^j))$.

- $i < j \leq l(\pi_n^i)$

  Then $n+1$ is inserted to the right of n giving

$$\pi_{n,n+1}^{i,j} = (p_1, \ldots, p_{i-1}, n, p_i, \ldots, p_{j-2}, n+1, p_{j-1}, \ldots, p_{n-1})$$

  Note that $n+1$ is inserted right behind p_{j-2}, and not p_{j-1}, because the position to the right of p_{j-2} is the j-th active site in π_n^i. The number of active sites in $\pi_{n,n+1}^{i,j}$ equals the number of active sites in $\pi_n^{j-1} = (p_1, \ldots, p_{j-2}, n, p_{j-1}, \ldots, p_{n-1})$ plus the additional active site next to entry n: $l(\pi_{n,n+1}^{i,j}) = l(\pi_n^{j-1}) + 1$.

In summary, we have obtained the number of active sites in a 1324-avoiding permutation of length $n+1$ in terms of the number of active sites in 1324-avoiding permutations of length n:

$$l(\pi_{n,n+1}^{i,j}) = \begin{cases} l(\pi_n^i) + 1 & \text{if } j = 1, \\ \min(i+1, l(\pi_n^j)) & \text{if } 2 \leq j \leq i, \\ l(\pi_n^{j-1}) + 1 & \text{if } i < j \leq l(\pi_n^i). \end{cases}$$

Clearly, the values $l(\pi_{n,n+1}^{i,j})$, $1 \leq j \leq l(\pi_n^i)$, depend on i and the values $l(\pi_n^j)$, $1 \leq j \leq l(\pi_n^i)$. Hence, if we assign label $(s(\pi), i)$, where $s(\pi) = \langle l(\pi_n^1) \ldots l(\pi_n^{l(\pi)}) \rangle$, to each π_n^i, for $1 \leq i \leq l(\pi)$, then the label of $\pi_{n,n+1}^{i,j}$ is completely determined by the label of its parent, π_n^i. More precisely, the label of $\pi_{n,n+1}^{i,j}$ is $(s(\pi_n^i), j)$; the sequence $s(\pi_n^i) = \langle l(\pi_{n,n+1}^{i,1}) \ldots l(\pi_{n,n+1}^{i,l(\pi_n^i)}) \rangle$ is given by the succession rule $s(\pi_n^i) = f(\langle l(\pi_n^1) \ldots l(\pi_n^{l(\pi)}) \rangle, i)$, where f is the function defined in (2). The root of the tree has the label $(\langle 2 \rangle, 1)$, which represents the unique permutation of length 1. This completes the proof of the lemma. $\square$

We next prove Theorem 1. Let T be the generating tree for 1324-avoiding permutations.

Proof. Let $d[(\langle a_1 \ldots a_m \rangle, i), n]$ be the number of 1324-avoiding permutations on the n-th level of the subtree of T, rooted at (the node with label) $(\langle a_1 \ldots a_m \rangle, i)$. Then,

$$d[(\langle a_1 \ldots a_m \rangle, i), n] = \begin{cases} 1 & \text{if } n = 0, \\ \sum_{j=1}^{a_i} d[(\langle b_1 \ldots b_{a_i} \rangle, j), n-1] & \text{if } n = 0. \end{cases}$$

Note that $d[(\langle a_1 \ldots a_m \rangle, i), 1] = \sum_{j=1}^{a_i} d[(\langle b_1 \ldots b_{a_i} \rangle, j), 0] = a_i$, since $d[(\langle b_1 \ldots b_{a_i} \rangle, j), 0] = 1$.

Let $g(\langle a_1 \ldots a_m \rangle, n)$ be the number of 1324-avoiding permutations on the n-th level of the subforest of T, which consists of trees whose roots are $(\langle a_1 \ldots a_m \rangle, i)$, $1 \leq i \leq m$. Then,

$$g(\langle a_1 \ldots a_m \rangle, n) = \sum_{i=1}^{m} d[(\langle a_1 \ldots a_m \rangle, i), n] = \sum_{i=1}^{m} \sum_{j=1}^{a_i} d[(f(\langle a_1 \ldots a_m \rangle, i), j), n-1]$$
$$= \sum_{i=1}^{m} g(f(\langle a_1 \ldots a_m \rangle, i), n-1).$$

□

3 Concluding remarks

Theorem 1 provides a recurrence formula for the number of 1324-avoiding permutations, which, with the help of a computer, gives values of $s_n(1324)$ up to $n = 20$ [SPBC96]. Figure 2 shows a simple Maple code that directly corresponds to Theorem 1; the procedure `count1324` counts the number of all 1324-avoiding permutations of length `n`, and the procedure `g` corresponds to g, with inlined f.

Note that `g` has `option remember` modifier. It instructs Maple to use memoization [Bel57, Mic68] for `g`. Namely, Maple maintains a table of the input pairs `s` and `n` and corresponding values for `g`. Before computing the value for some pair, Maple first checks if that pair is already in the table. If so, Maple immediately returns the value; otherwise, it computes the value and stores the pair and the value in the table. The use of memoization significantly reduces time for computing the values of `g` for larger `n`. However, the memoization table requires space. On machines on which we used Maple, it ran out of memory when `n` was 15. We rewrote the code from Figure 2 in Java to speed up the computation and to reduce the memory consumption. The Java code uses a more compact representation of sequences of small numbers. It also has a selective memoization that stores in the table only the input pairs (and their corresponding values) for which `g` is likely to be invoked several times. We ran the Java code on the Sun JVM version 1.3.0 running under Linux on a 2GHz Pentium IV machine with 2GB of memory. Computing the number of 1324-avoiding permutations of length 20 took about 5 hours.

Although we have obtained a recurrence formula for the number of all 1324-avoiding permutations, we do not have a closed form for $s_n(1324)$. The occurrence of the min function in the definition of f, together with the fact that the length of the sequences assigned to nodes of the generating tree increase with the node level in the tree, complicate any attempt to obtain a closed formula. But, the formula may help finding the asymptotic growth of $s_n(1324)$.

In 1990, Stanley and Wilf conjectured that $s_n(\tau) < (c(\tau))^n$, where $c(\tau)$ is a constant. This conjecture clearly holds for patterns of length 3. Results of Bóna and Regev [Bón97a, Reg81] imply that $s_n(1342) < 8^n$ and $s_n(1234) < 9^n$, these bounds being asymptotically tight. Moreover, Bóna [Bón97b] proves that $s_n(1324)$ is asymptotically larger than $s_n(1234)$, and sketches an argument to prove that $s_n(1324) < 36^n$, this bound almost certainly not being tight. His techniques use the idea of dividing permutations into strong classes.

```
count1324 := proc(n)
     return g([1], n);
end:

g := proc(s, n) option remember;
     local i, j, sum, sNext;
     if (n = 1) then
         return convert(s, ‘+‘);
     fi;

     sum := 0;
     for i from 1 to nops(s) do
         sNext := s[i] + 1;
         for j from 2 to i do
             sNext := sNext, ‘min‘(i + 1, s[j]);
         od;
         for j from i + 1 to s[i] do
             sNext := sNext, s[j - 1] + 1;
         od;
         sum := sum + g([sNext], n - 1);
     od;
     return sum;
end:
```

Figure 2: The Maple code for counting 1324-avoiding permutations

n	$s_n(1324)$
0	1
1	1
2	2
3	6
4	23
5	103
6	513
7	2,762
8	15,793
9	94,776
10	591,950
11	3,824,112
12	25,431,452
13	173,453,058
14	1,209,639,642
15	8,604,450,011
16	62,300,851,632
17	458,374,397,312
18	3,421,888,118,907
19	25,887,131,596,018
20	198,244,731,603,623

Figure 3: The number of 1324-avoiding permutations for length up to 20

Definition 5. Two permutations π and σ are said to be in the same *strong* class if the left-to right minima of π are the same as those of σ and they occur in the same position; and the same is true also of the right-to-left maxima.

Strong classes are denoted by specifying the positions of their minima and maxima and writing a '$*$' in the other positions. For example, $7{*}5{*}3{*}1{*}13{*}11{*}9$ denotes the strong class whose left-to-right minima are 7,5,3,1 (at positions 1,3,5,7) and right-to-left maxima are 13,11,9 (at positions 9,11,13). This particular strong class is, in fact, the class $S_{4,3}$ where, in general, $S_{l,r}$ is the strong class whose left-to-right minima $2l+1, 2l-1, \dots$ occur at the odd numbered positions followed by the right-to-left maxima $2(l+r)-1, 2(l+r)-3, \dots$ occurring at the remaining odd numbered positions.

Bóna showed that there are at most 9^n non-empty strong classes and sketched a proof that each one contains at most 4^n 1324-avoiding permutations. From our experiments with the Java applet [Str03] provided by Atkinson and his group we conjecture with some confidence that

Conjecture 6. *If $n = 2(l+r)-1$, the strong class $S_{l,r}$ contains more 1324-avoiding permutations than any other strong class with l left-to-right minima and r right-to-left maxima. Furthermore, the strong class $S_{r,r}$ contains more 1324-avoiding permutations than any other strong class of that length.*

We actually know the exact formula for $|S_{l,r}|$.

Proposition 7. $|S_{l,r}| = \binom{l+r-1}{l-1}$.

Proof. Let $n = 2k+1$. Let $a_l, \ldots, a_1$ be the left-to-right minima, and $b_r, \ldots, b_1$ be the right-to-left maxima. Here, the sequence $a_1, \ldots, a_l, b_1, \ldots, b_r$ is actually the sequence $1, 3, \ldots, n$. Let $\sigma \in S_{l,r}$. It is easy to see that: 1) if $k+1$ occurs to the left of $b_r = n$, then $k+1$ has to be the second entry of σ; and 2) if $k+1$ occurs to the right of $a_1 = 1$, then $k+1$ has to be the next-to-last entry of σ. Hence, 1324-avoiding permutations in $S_{l,r}$ fall into two categories: the ones with $\sigma(2) = k+1$ and the ones with $\sigma(n-1) = k+1$. We map each $\sigma = (k, k+1, k-1, \gamma) \in S_{l,r}$ to $\sigma' = (k-1, \gamma') \in S_{l-1,r}$, and vice versa, where γ' is obtained from γ by reducing all the entries of γ that are greater than $k+1$ by 2. Therefore, 1324-avoiding permutations in $S_{l,r}$ with $k+1$ as the second entry are in one-to-one correspondence with 1324-avoiding permutations in $S_{l-1,r}$. Similarly, 1324-avoiding permutations in $S_{l,r}$ with $k+1$ as the next-to-last entry are in one-to-one correspondence with 1324-avoiding permutations in $S_{l,r-1}$. Thus, $|S_{l,r}| = |S_{l-1,r}| + |S_{l,r-1}|$, completing the proof by induction. □

Since $\binom{2r-1}{r-1} < 2^{n/2}$, the conjecture would prove that $s_n(1324) < (9\sqrt{2})^n$, which would be a considerable improvement on Bóna's bound. It remains plausible that $s_n(1324) < 9^n$.

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